ON A CONVOLUTION EQUATION AND ITS APPLICATIONS TO CHARACTERIZATION PROBLEMS

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Recently the integrated Cauchy functional equation

$$f(x) = \int_0^\infty f(x + y)\, d\sigma(y), \quad \forall\ x \geqslant 0,$$

and its applications to characterization problems have been under intensive investigation. We extend the discussions from the domain $R_+ = [0, \infty]$ to a locally compact semigroup S, and characterize the nonnegative solutions f by the extreme point method. The special case with $S = R_+^2$, and its applications to the randomized lack of memory property and generalized stable laws are discussed.

1. INTRODUCTION

Many problems in characterizations of probability distributions can be reduced to solving the integrated Cauchy functional equation (ICFE (σ)) on $[0, \infty)$

$$f(x) = \int_0^\infty f(x + y)\, d\sigma(y), \quad \forall\ x \geqslant 0, \tag{1.1}$$

where σ is a given positive Borel measure on $R_+ = [0, \infty)$. (see e.g. [s], [9]). Any nonnegative locally integrable solution f of the above ICFE (σ) is shown to be of the form $f(x) = p(x)\, e^{-\alpha x}$, where p is a periodic function with every $y \in \operatorname{supp} \sigma$ as period, and α satisfies

$$\int_0^\infty e^{-\alpha x}\, d\sigma(x) = 1.$$

A real-variable proof of the above was given by Lau and Rao [5] in 1982, free of all superfluous (previous) assumptions, and later simplified by Ramachandran [8].

The ICFE(σ) is closely related to the Choquet-Deny convolution equation

$$f(x) = \int_G f(x - y) \, d\sigma(y), \quad \forall \, x \in G, \tag{1.2}$$

where G is a locally compact abelian group. The bounded solutions to the equation (1.2) with σ a probability measure were characterized by Choquet and Deny [1], and found applications in renewal theory. Using the extreme point method (the Choquet theorem), Deny [3] derived an integral representation for nonnegative (possibly unbounded) solutions f of the equation (1.2):

$$f(x) = \int_{E(\sigma)} g(x) \, dP(g),$$

where $E(\sigma)$ is the set of scalar multiples of exponential functions g on G with

$$\int_G g(-y) \, d\sigma(y) = 1,$$

and P is a probability measure on $E(\sigma)$.

It is natural to attempt to extend (1.1) to semigroups. Here we consider the ICFE(σ) on a semigroup S:

$$f(x) = \int_S f(x + y) \, d\sigma(y), \quad \forall \, x \in S, \tag{1.3}$$

Davies and Shanbhag [2] proved by martingale arguments an extension of Deny's theorem when S is a suitable subsemigroup of a separable, metrizable, locally compact abelian group. They also pointed out by an example that the solutions to the equation (1.3) are more complicated, and that the proof of Deny in [3] does not generalize automatically.

In this note, we obtain in our main theorems an integral representation of the general nonnegative solutions f of the ICFE(σ) (1.3) on locally compact semigroups. The approach is by introducing a "skew" convolution and applying the Choquet theorem as in Deny [3]. Our main theorems are illustrated by several examples on $\mathbf{R}_+^2$. Applications to bivariate characterization problems such as randomized lack of memory property and generalized stable laws are also discussed. The proofs will appear in [4] and [7].

2. THE THEOREMS

For simplicity we let S be a subsemigroup of a separable, metrizable,

locally compact abelian group G, and let S have nonvoid interior S^0 (the more-general case can be found in [6] and [7]). Then there is a translation invariant measure ω on S.

A function $g : S \to \mathbf{R}$ is called *exponential* if $g \neq 0$ is a continuous function, and $g(x + y) = g(x) g(y)$ for all $x, y \in S$. Let σ be a Borel measure on S. An exponential function g is said to be σ-*harmonic* if g satisfies

$$\int_S g(y) \, d\sigma(y) = 1.$$

Let $E(\sigma)$ denote the set of all λg, where $\lambda > 0$ and g is a σ-harmonic exponential function on S. Also let

$$E_0(\sigma) = \{g \in E(\sigma) : g(x) > 0 \text{ for some } x \in S^0\}.$$

The main theorems are:

THEOREM 2.1. If $S = S(\sigma)$, the closed subsemigroup generated by the support of σ, and if $f \geq 0$ is a ω-locally-integrable solution of the ICFE(σ) (1.3), then for each $y \in S^0$,

$$f(\cdot + y) = \int_{E_0(\sigma)} g(y) g(\cdot) \, dP(g), \quad [\omega]\text{-a.e. on } S^0,$$

where P is a probability measure on $E_0(\sigma)$.

If further $\overline{S^0} = S$, and f is continuous, then for all $x \in S$,

$$f(x) = \int_{E_0(\sigma)} g(x) \, dP(g). \tag{2.1}$$

Unlike in the group case, the condition $S = S(\sigma)$ is too restrictive on σ for most of the applications. We introduce what we call the *component-generating property* and obtain:

THEOREM 2.2. Suppose $S(\sigma) = \overline{S^0}$, and $S(\sigma)$ satisfies the condition that for each open and closed subsemigroup T in $S(\sigma)$, $(S(\sigma) - T) \cap S$ is dense in S (which is called the *component-generating property* of S). Then any nonnegative continuous solution f of the ICFE(σ) (1.3) admits the integral representation (2.1).

The following is also useful in reducing the problem to a simpler one.

PROPOSITION 2.3. Let S' be a subsemigroup of S. Suppose $S(\sigma) \subseteq S'$, and there exists $D \subseteq S$ such that $\{s + S' : s \in D\}$ is a family of disjoint sets whose union is S; then the solution f to the ICFE(σ) (1.3) is of the form

$$f(s + x) = p(s) \, g(x), \quad s \in D, x \in S',$$

where g satisfies

$$g(x) = \int_{S'} g(x + y)\, d\sigma(y), \quad \forall\ x \in S'.$$

The proofs of the main theorems (in a more comprehensive setting) are by the extreme point method, (see [7] for details). We illustrate the theorems by some examples.

EXAMPLE 2.4. Let $S = \mathbf{R}_+$, then the solution f of (1.1) mentioned in the introduction can be deduced by letting $T = S(\sigma)$,

$$S' = (\mathbf{R}_+ \setminus \{0\}) \cap (T - T).$$

If T is nonarithmetic, apply Theorem 2.2 to S'; otherwise apply Theorem 2.2 and Proposition 2.3 together to S'.

EXAMPLE 2.5. Let $S = \mathbf{R}_+^2$. If $S(\sigma)$ has the component-generating property, then Theorem 2.2 implies that any locally integrable solution f of the ICFE(σ) (1.3) is given by

$$f(x, y) = \int_A \exp\left(-(\alpha x + \beta y)\right) d\tau(\alpha, \beta), \quad [\omega]\text{-a.e. for } (x, y) \in S,$$

where A is the set of (α, β) in $\mathbf{R}^2$ satisfying

$$\int_{\mathbf{R}_+^2} \exp(-(\alpha s + \beta t))\, d\sigma(s, t) = 1, \tag{2.2}$$

and τ is a positive regular measure on A.

EXAMPLE 2.6. Suppose $S = \mathbf{R}_+^2$, and supp $\sigma \subseteq \{(x, x) : x \geqslant 0\}$ is non-arithmetic. We let $S' = $ supp σ, and

$$D = (\mathbf{R}_+ \times \{0\}) \cup (\{0\} \times \mathbf{R}_+).$$

By Proposition 2.3 and Example 2.4, we conclude that any nonnegative solution f is of the form

$$f(x, y) = p_1(x - y) \exp(-\alpha y), \quad x \geqslant y \geqslant 0;$$
$$= p_2(y - x) \exp(-ax), \quad y \geqslant x \geqslant 0,$$

where $\int_0^\infty \exp(-\alpha t)\, d\sigma(t) = 1$, and p_1, p_2 are nonnegative functions on $\mathbf{R}_+$.

EXAMPLE 2.7. Let $S = \mathbf{R}_+^2$, $S_1 = \{0\} \times \mathbf{R}_+$, $S_2 = (1, \infty) \times \mathbf{R}_+$, and let σ be such that supp $\sigma = S_1 \cup S_2$ (Davies and Shanbhag [2]). In this case $S(\sigma)$ does *not* have the component-generating propery in $\mathbf{R}_+^2$. Note that

$E_0(\sigma)$ consists of exponential functions of the form $ce^{-(\alpha x+\beta y)}$, $(x, y) \in S$, with

$$\int_{S_1} \exp(-\beta y) \, d\sigma(0, y) + \int_{S_2} \exp(-(\alpha x + \beta y)) \, d\sigma(x, y) = 1,$$

and $E(\sigma)$ is the union of $E_0(\sigma)$ and the set of functions g with

$$g(x, y) = ce^{-\gamma y}, \quad \text{if } x = 0,$$
$$= 0, \quad \text{if } x \neq 0,$$

where $\int_0^\infty e^{-\gamma t} \, d\sigma(0, t) = 1$. We conclude from Theorem 2.1 that a continuous solution f satisfies

$$f(x, y) = \int_A \exp(-(\alpha x + \beta y)) \, d\tau(\alpha, \beta), \quad x \geq 2, y > 0,$$

where A is the set of (α, β) satisfying (2.2), and τ is a positive measure on A. A further inspection shows that f has the representation

$$f(x, y) = ce^{-\gamma y} + \int_A \exp(-(\alpha x + \beta y)) \, d\tau(\alpha, \beta), \quad \forall \, (x, y) \in S(\sigma).$$

3. APPLICATIONS TO CHARACTERIZATION PROBLEMS

Randomized Lack of Memory Property

Let X denote the life-time of a system. X is said to have the *lack of memory property* if

$$P(X > x + t / X > t) = P(X > x), \quad \forall \, t, x \geq 0.$$

A more general model is that

$$P(X > x + T / X > T) = P(X > t), \quad \forall \, x \geq 0,$$

where $T \geq 0$ is a random checking time. We call this the *randomized lack of memory property* (rLMP). It is known that if T is nonarithmetic, then X has rLMP if and only if X has an exponential distribution.

For the bivariate case, we have two extensions:

THEOREM 3.1. Suppose (X, Y), and (T_1, T_2) are nonnegative random vectors, and satisfy

$$P(X > x + T_1, Y > y + T_2 / X > T_1, Y > T_1) = P(X > x, Y > y),$$

$$\forall \, (x, y) \geq 0.$$

If the subsemigroup generated by the essential range of (T_1, T_2) has the component-generating property in $\mathbf{R}_+^2$, then the distribution of (X, Y) is a

mixture of exponential distributions with survival functions (see the definition below) of the form

$$\exp\left(-(\alpha x + \beta y)\right), \quad x, y \geq 0,$$

where $\alpha, \beta \geq 0$ satisfy

$$\int_{\mathbf{R}_+^2} \exp\left(-(\alpha s + \beta t)\right) \, d\sigma(s, t) = 1,$$

and $d\sigma(s, t) = \{P(X > s, Y > t)\}^{-1} \, d\mathrm{F}_{(T_1, T_2)}(s, t)$.

THEOREM 3.2.   Suppose (X, Y), T are nonnegative random variables satisfying

$$P(X > x + T, Y > y + T / X, Y > T) = P(X > x, Y > y),$$

$$\forall \ (x, y) \geq 0.$$

If T is nonarithmetic, then the survival function

$$G(x, y) = P(X > x, Y > y)$$

is of the form

$$G(x, y) = G_1(x - y) \, e^{-\alpha y}, \quad x \geq y \geq 0,$$

$$= G_2(y - x) \, e^{-\alpha x}, \quad y \geq x \geq 0,$$

where $\displaystyle\int_0^\infty e^{-\alpha t} \, d\sigma(t) = 1$, $d\sigma(t) = \{P(X > s, Y > t)\}^{-1} \, dF_T(t)$, and G_1, G_2 are the survival functions of X and Y, respectively.

Notice that if X and Y in Theorem 3.2 are exponentially distributed, then (X, Y) has the bivariate exponential distribution of Marshall and Olkin. Both Theorem 3.1 and 3 2 can be easily extended to the multivariate case, which discussion will appear elsewhere. We remark that the multivariate analog of Theorem 3.1 generalizes a result in Davies and Shanbhag [2], where the essential range of the random checking time (T_1, T_2) is assumed to be $\mathbf{R}_+^2$.

Generalized Stable Laws

A random variable X is said to obey a *generalized stable law* if its characteristic function ϕ satisfies

$$\phi(t) = \prod_j \phi(\beta_j t)^{\gamma_j}, \quad \forall \ t \in \mathbf{R},$$

where $|\beta_j| < 1$ and $\gamma_j > 0$. The bivariate analog is

$$\phi(t_1, t_2) = \prod_j \phi(\beta_j(t_1, t_2))^{\gamma_j}, \quad \forall \ t_1, t_2 \in \mathbf{R}, \tag{3.1}$$

If we let

$$\Psi(t_1, t_2) = |\phi(t_1, t_2)|^2, \quad \text{and} \quad G(x, y) = -\log \Psi(e^{-x}, e^{-y}),$$

then the equation (3.1) implies that

$$G(x, y) = \int_0^\infty G(x + t, y + t)\, d\sigma(t),$$

with supp $\sigma = \{-\log |\beta_j| : j = 1, 2, \ldots\}$ and $\sigma\{-\log |\beta_j|\} = \gamma_j > 0$. It follows that (provided that σ is nonarithmetic)

$$\Psi(t_1, t_2) = \exp\{-c_1 |t_1|^\alpha - c_2 |t_2|^\alpha - |t_1 t_2|^{\alpha/2} h(t_1, t_2)\},$$

where α is determined by $\sum_j \gamma_j |\beta_j|^\alpha = 1$, and h is a continuous function which satisfies $h(st_1, st_2) = h(t_1, t_2)$ for all $s \geq 0$. Hence we have

THEOREM 3.3. Suppose that $\{-\log |\beta_j| : j = 1, 2, \ldots\}$ is nonarithmetic with at least one $\beta_j > 0$. If the characteristic function ϕ of a nondegenerate random vector X satisfies equation (3.1), then $0 < \alpha \leq 2$, and

(1) If $\alpha = 2$, then X has a bivariate normal distribution,

(2) If $1 < \alpha < 2$, then

$$\phi(t_1, t_2) = \exp\{-c_1 |t_1|^\alpha - c_2 |t_2|^\alpha - |t_1 t_2|^{\alpha/2}$$
$$h(t_1, t_2) + i(m_1 t_1 + m_1 t_1)\},$$

where $m_1, m_2 \neq 0$, only if $\sum_j \gamma_j \beta_j = 1$,

(3) If $0 < \alpha \leq 1$, then

$$\phi(t_1, t_2) = \exp\{-c_1 |t_1|^\alpha - c_2 |t_2|^\alpha - |t_1 t_2|^{\alpha/2} h(t_1, t_2)\}.$$

Theorem 3.3 has been generalized to multivariate distributions in Gupta, Nyugen and Zeng [4].

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